CS 205b / CME 306

Application Track

Homework 8

1 Overview

It is shown in class that for incompressible flow the inviscid 1D Euler equations decouple to:

$$\rho_t + u\rho_x = 0 \qquad u_t + \frac{p_x}{\rho} = 0 \qquad e_t + ue_x = 0$$

In this homework assignment, you will extend this to the 3D case.

This assignment is an algebraic exercise that asks you to manipulate one set of equations (compressible flow) into another set of equations (incompressible flow) using additional assumptions. The difficulty of these manipulations and the work required to complete them depends on the notation used. The primary purpose of this assignment is to see how the incompressible equations are obtained from the compressible ones. A secondary (but optional) purpose of this assignment is to experiment with different notation.

The assignment is expressed using four different types of notation. You should choose one of the four notational styles and use it consistently.

2 Calculus Notation

The 3D Euler equations are given by

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ (E+p)u \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vw \\ (E+p)v \end{pmatrix}_y + \begin{pmatrix} \rho w \\ \rho uw \\ \rho vw \\ \rho w^2 + p \\ (E+p)w \end{pmatrix}_z = 0 \tag{1}$$

where ρ is the density, $\mathbf{u} = (u, v, w)$ are the velocities, E is the total energy per unit volume and p is the pressure. The total energy is the sum of the internal energy and the kinetic energy.

$$E = \rho e + \rho(u^2 + v^2 + w^2)/2$$

where e is the internal energy per unit mass. The assumption of incompressiblity gives

$$u_x + v_y + w_z = 0, \tag{2}$$

Show that in 3D the inviscid Euler equations with the assumption of incompressible flow decouple to:

$$\begin{aligned}
\rho_t + u\rho_x + v\rho_y + w\rho_z &= 0 \\
u_t + uu_x + vu_y + wu_z + \frac{p_x}{\rho} &= 0 \\
v_t + uv_x + vv_y + wv_z + \frac{p_y}{\rho} &= 0 \\
w_t + uw_x + vw_y + ww_z + \frac{p_z}{\rho} &= 0 \\
e_t + ue_x + ve_y + we_z &= 0
\end{aligned}$$

3 Vector Calculus Notation

The 3D Euler equations take the form

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0 \qquad (\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u}\mathbf{u}^T) + \nabla p = 0 \qquad E_t + \nabla \cdot ((E + p)\mathbf{u}) = 0$$

The break down of total energy and the incompressibility assumption take the form

$$E = \rho \left(e + \frac{1}{2} \|\mathbf{u}\|^2 \right) \qquad \nabla \cdot \mathbf{u} = 0$$

Show that in 3D the inviscid Euler equations with the assumption of incompressible flow decouple to:

$$\rho_t + \mathbf{u} \cdot \nabla \rho = 0 \qquad \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho} \nabla p = 0 \qquad e_t + \mathbf{u} \cdot \nabla e = 0$$

Note that the second equation differs from the version in the lecture notes. The version in the lecture notes is technically incorrect but reflects the way the equation is often written. It should not be used here, since the equation can only be obtained by making a mistake or by intentionally introducing the notational abuse. The difference between the two is subtle, but only one of them will produce correct results.

Mass:

$$\begin{aligned}
0 &= \rho_t + \nabla \cdot (\rho \mathbf{u}) \\
&= \rho_t + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} \\
&= \rho_t + \mathbf{u} \cdot \nabla \rho
\end{aligned}$$

Momentum:

$$\begin{aligned}
0 &= (\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u}\mathbf{u}^T) + \nabla p \\
&= \rho_t \mathbf{u} + \rho \mathbf{u}_t + (\mathbf{u} \cdot \nabla \rho)\mathbf{u} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \rho \mathbf{u}(\nabla \cdot \mathbf{u}) + \nabla p \\
&= \rho \mathbf{u}_t + (\rho_t + \mathbf{u} \cdot \nabla \rho)\mathbf{u} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p \\
&= \rho \mathbf{u}_t + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p \\
0 &= \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho} \nabla p
\end{aligned}$$

Energy:

$$
\begin{aligned}
0 &= E_t + \nabla\cdot((E+p)\mathbf{u}) \\
&= E_t + \mathbf{u}\cdot\nabla(E+p) + (E+p)\nabla\cdot\mathbf{u} \\
&= \left(\rho\left(e+\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right)\right)_t + \mathbf{u}\cdot\nabla\left(\rho\left(e+\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right)\right) + \mathbf{u}\cdot\nabla p \\
&= \rho\left(e+\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right)_t + \rho_t\left(e+\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) + \mathbf{u}\cdot\nabla\rho\left(e+\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) + \rho\mathbf{u}\cdot\nabla\left(e+\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) + \mathbf{u}\cdot\nabla p \\
&= \rho\left(e+\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right)_t + (\rho_t + \mathbf{u}\cdot\nabla\rho)\left(e+\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) + \rho\mathbf{u}\cdot\nabla\left(e+\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) + \mathbf{u}\cdot\nabla p \\
&= \rho\left(e+\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right)_t + \rho\mathbf{u}\cdot\nabla\left(e+\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) + \mathbf{u}\cdot\nabla p \\
&= \rho e_t + \rho\mathbf{u}\cdot\nabla e + \rho\mathbf{u}\cdot\mathbf{u}_t + \rho\mathbf{u}\cdot((\mathbf{u}\cdot\nabla)\mathbf{u}) + \mathbf{u}\cdot\nabla p \\
&= \rho e_t + \rho\mathbf{u}\cdot\nabla e + \mathbf{u}\cdot(\rho\mathbf{u}_t + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p) \\
&= \rho e_t + \rho\mathbf{u}\cdot\nabla e
\end{aligned}
$$

4 Classical Cartesian Tensor Calculus Notation

The 3D Euler equations take the form

$$\frac{\partial\rho}{\partial t} + (\rho u_i)_{,i} = 0 \qquad \frac{\partial}{\partial t}(\rho u_i) + (\rho u_i u_k)_{,k} + p_{,i} = 0 \qquad \frac{\partial E}{\partial t} + ((E+p)u_i)_{,i} = 0$$

The break down of total energy and the incompressibility assumption take the form

$$E = \rho\left(e + \frac{1}{2}u_i u_i\right) \qquad u_{i,i} = 0.$$

Show that in 3D the inviscid Euler equations with the assumption of incompressible flow decouple to:

$$\frac{\partial\rho}{\partial t} + u_i\rho_{,i} = 0 \qquad \frac{\partial u_i}{\partial t} + u_k u_{i,k} + \frac{1}{\rho}p_{,i} = 0 \qquad \frac{\partial e}{\partial t} + u_i e_{,i} = 0$$

Mass:

$$
\begin{aligned}
0 &= \rho_t + (\rho u_i)_{,i} \\
&= \rho_t + \rho_{,i}u_i + \rho u_{i,i} \\
&= \rho_t + \rho_{,i}u_i
\end{aligned}
$$

Momentum:

$$\begin{aligned}
0 &= (\rho u_i)_t + (\rho u_i u_k)_{,k} + p_{,i} \\
&= \rho(u_i)_t + \rho_t u_i + \rho_{,k} u_i u_k + \rho u_{i,k} u_k + \rho u_i u_{k,k} + p_{,k}\delta_{ik} \\
&= \rho(u_i)_t + (\rho_t + \rho_{,k} u_k) u_i + \rho u_{i,k} u_k + p_{,i} \\
&= \rho(u_i)_t + \rho u_{i,k} u_k + p_{,i} \\
0 &= (u_i)_t + u_{i,k} u_k + \frac{1}{\rho} p_{,i}
\end{aligned}$$

Energy:

$$\begin{aligned}
0 &= E_t + ((E+p)u_i)_{,i} \\
0 &= E_t + (E_{,i} + p_{,i})u_i + (E+p)u_{i,i} \\
0 &= \left(\rho\left(e + \frac{1}{2}u_k u_k\right)\right)_t + \left(\rho\left(e + \frac{1}{2}u_k u_k\right)\right)_{,i} u_i + p_{,i} u_i \\
0 &= \rho\left(e + \frac{1}{2}u_k u_k\right)_t + \rho_t\left(e + \frac{1}{2}u_k u_k\right) + \rho_{,i}\left(e + \frac{1}{2}u_k u_k\right)u_i + \rho\left(e + \frac{1}{2}u_k u_k\right)_{,i} u_i + p_{,i} u_i \\
0 &= \rho\left(e + \frac{1}{2}u_k u_k\right)_t + (\rho_t + \rho_{,i} u_i)\left(e + \frac{1}{2}u_k u_k\right) + \rho\left(e + \frac{1}{2}u_k u_k\right)_{,i} u_i + p_{,i} u_i \\
0 &= \rho\left(e + \frac{1}{2}u_k u_k\right)_t + \rho\left(e + \frac{1}{2}u_k u_k\right)_{,i} u_i + p_{,i} u_i \\
0 &= \rho e_t + \rho u_k (u_k)_t + \rho e_{,i} u_i + \rho u_k u_{k,i} u_i + p_{,i} u_i \\
0 &= \rho e_t + \rho e_{,i} u_i + \rho u_i (u_i)_t + \rho u_i u_{i,k} u_k + p_{,i} u_i \\
0 &= \rho e_t + \rho e_{,i} u_i + (\rho (u_i)_t + \rho u_{i,k} u_k + p_{,i}) u_i \\
0 &= \rho e_t + \rho e_{,i} u_i \\
0 &= e_t + e_{,i} u_i
\end{aligned}$$

5 Classical Tensor Calculus Notation

The 3D Euler equations take the form

$$\frac{\partial \rho}{\partial t} + (\rho u^i)_{,i} = 0 \qquad \frac{\partial}{\partial t}(\rho u^i) + (\rho u^i u^k)_{,k} + g^{ik} p_{,k} = 0 \qquad \frac{\partial E}{\partial t} + ((E+p)u^i)_{,i} = 0$$

The break down of total energy and the incompressibility assumption take the form

$$E = \rho\left(e + \frac{1}{2} g_{ik} u^i u^k\right) \qquad u^i_{,i} = 0.$$

Show that in 3D the inviscid Euler equations with the assumption of incompressible flow decouple to:

$$\frac{\partial \rho}{\partial t} + u^i \rho_{,i} = 0 \qquad \frac{\partial u^i}{\partial t} + u^k u^i_{,k} + \frac{1}{\rho} g^{ik} p_{,k} = 0 \qquad \frac{\partial e}{\partial t} + u^i e_{,i} = 0$$

Mass:

$$\begin{aligned}
0 &= \rho_t + (\rho u^i)_{,i} \\
&= \rho_t + \rho_{,i} u^i + \rho u^i_{,i} \\
&= \rho_t + \rho_{,i} u^i
\end{aligned}$$

Momentum:

$$\begin{aligned}
0 &= (\rho u^i)_t + (\rho u^i u^k)_{,k} + g^{ik} p_{,k} \\
&= \rho (u^i)_t + \rho_t u^i + \rho_{,k} u^i u^k + \rho u^i_{,k} u^k + \rho u^i u^k_{,k} + g^{ik} p_{,k} \\
&= \rho (u^i)_t + (\rho_t + \rho_{,k} u^k) u^i + \rho u^i_{,k} u^k + g^{ik} p_{,k} \\
&= \rho (u^i)_t + \rho u^i_{,k} u^k + g^{ik} p_{,k} \\
&= (u^i)_t + u^i_{,k} u^k + \frac{1}{\rho} g^{ik} p_{,k}
\end{aligned}$$

Energy:

$$\begin{aligned}
0 &= E_t + ((E+p)u^i)_{,i} \\
0 &= E_t + (E_{,i} + p_{,i}) u^i + (E+p) u^i_{,i} \\
0 &= \left(\rho\left(e + \frac{1}{2} g_{mk} u^m u^k\right)\right)_t + \left(\rho\left(e + \frac{1}{2} g_{mk} u^m u^k\right)\right)_{,i} u^i + p_{,i} u^i \\
0 &= \rho\left(e + \frac{1}{2} g_{mk} u^m u^k\right)_t + \rho_t\left(e + \frac{1}{2} g_{mk} u^m u^k\right) + \rho_{,i}\left(e + \frac{1}{2} g_{mk} u^m u^k\right) u^i + \rho\left(e + \frac{1}{2} g_{mk} u^m u^k\right)_{,i} u^i + p_{,i} u^i \\
0 &= \rho\left(e + \frac{1}{2} g_{mk} u^m u^k\right)_t + (\rho_t + \rho_{,i} u^i)\left(e + \frac{1}{2} g_{mk} u^m u^k\right) + \rho\left(e + \frac{1}{2} g_{mk} u^m u^k\right)_{,i} u^i + p_{,i} u^i \\
0 &= \rho\left(e + \frac{1}{2} g_{mk} u^m u^k\right)_t + \rho\left(e + \frac{1}{2} g_{mk} u^m u^k\right)_{,i} u^i + p_{,i} u^i \\
0 &= \rho e_t + \frac{1}{2} \rho g_{mk} (u^m)_t u^k + \frac{1}{2} \rho g_{mk} u^m (u^k)_t + \rho e_{,i} u^i + \frac{1}{2} \rho g_{mk} u^m_{,i} u^k u^i + \frac{1}{2} \rho g_{mk} u^m u^k_{,i} u^i + p_{,i} u^i \\
0 &= \rho e_t + \rho g_{mk} (u^m)_t u^k + \rho e_{,i} u^i + \rho g_{mk} u^m_{,i} u^k u^i + p_{,i} u^i \\
0 &= \rho e_t + \rho e_{,i} u^i + \rho g_{mn} (u^m)_t u^n + \rho g_{mn} u^m_{,i} u^n u^i + p_{,n} u^n \\
0 &= \rho e_t + \rho e_{,i} u^i + \left(\rho g_{mn} (u^m)_t + \rho g_{mn} u^m_{,i} u^i + p_{,n}\right) u^n \\
0 &= \rho e_t + \rho e_{,i} u^i + \rho g_{mn} \left((u^m)_t + u^m_{,i} u^i + \frac{1}{\rho} g^{mr} p_{,r}\right) u^n \\
0 &= \rho e_t + \rho e_{,i} u^i \\
0 &= e_t + e_{,i} u^i
\end{aligned}$$