CS 205b / CME 306

Application Track

Homework 3

1. A simple spring between particles at $\mathbf{x}_1$ and $\mathbf{x}_2$ in 3D can be defined by the equations

$$\mathbf{F} = m_1 \frac{d\mathbf{v}_1}{dt} \quad \mathbf{F} = -m_2 \frac{d\mathbf{v}_2}{dt} \quad \mathbf{u} = \frac{\Delta\mathbf{x}}{\|\Delta\mathbf{x}\|} \quad \mathbf{F} = -k_s\left(\frac{\|\Delta\mathbf{x}\|}{x_0} - 1\right)\mathbf{u} - k_d(\Delta\mathbf{v}\cdot\mathbf{u})\mathbf{u}$$
$$\mathbf{v}_1 = \frac{d\mathbf{x}_1}{dt} \quad \mathbf{v}_2 = \frac{d\mathbf{x}_2}{dt} \quad \Delta\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2 \quad \Delta\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2. \tag{1}$$

We would like to examine transformations under which these equations are invariant. That is, consider the new quantities obtained by applying transforms of the form

$$\begin{aligned}
\hat{\mathbf{x}}_1 &= \mathbf{A}_1\mathbf{x}_1 + \mathbf{b}_1 & \hat{\mathbf{v}}_1 &= \mathbf{A}_2\mathbf{v}_1 + \mathbf{b}_2 & \hat{\mathbf{F}} &= \mathbf{A}_3\mathbf{F} + \mathbf{b}_3 \\
\hat{\mathbf{x}}_2 &= \mathbf{A}_4\mathbf{x}_2 + \mathbf{b}_4 & \hat{\mathbf{v}}_2 &= \mathbf{A}_5\mathbf{v}_2 + \mathbf{b}_5 & & \\
\hat{m}_1 &= \alpha_3 m_1 + \beta_3 & \hat{m}_2 &= \alpha_4 m_2 + \beta_4 & \hat{t} &= \alpha_5 t + \beta_5 \\
\hat{x_0} &= \alpha_6 x_0 + \beta_6 & \hat{k_s} &= \alpha_1 k_s + \beta_1 & \hat{k_d} &= \alpha_2 k_d + \beta_2 \\
\hat{\Delta\mathbf{x}} &= \hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2 & \hat{\Delta\mathbf{v}} &= \hat{\mathbf{v}}_1 - \hat{\mathbf{v}}_2 & \hat{\mathbf{u}} &= \frac{\hat{\Delta\mathbf{x}}}{\|\hat{\Delta\mathbf{x}}\|}
\end{aligned}$$

where $\mathbf{A}_i$ are non-singular matrices with positive determinant and α_i are positive. All of the matrices $\mathbf{A}_i$, vectors $\mathbf{b}_i$, and scalars α_i and β_i are constants. That is, they do not depend on t, $\mathbf{x}_1$, $\mathbf{x}_2$, or any of the other quantities that occur in (1). We also require that these transformed quantities also satisfy

$$\hat{\mathbf{F}} = m\frac{d\hat{\mathbf{v}}_1}{d\hat{t}} \quad \hat{\mathbf{F}} = -\hat{m}\frac{d\hat{\mathbf{v}}_2}{d\hat{t}} \quad \mathbf{u} = \frac{\hat{\Delta\mathbf{x}}}{\|\hat{\Delta\mathbf{x}}\|} \quad \hat{\mathbf{F}} = -\hat{k_s}\left(\frac{\|\hat{\Delta\mathbf{x}}\|}{\hat{x_0}} - 1\right)\hat{\mathbf{u}} - \hat{k_d}(\hat{\Delta\mathbf{v}}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}}$$
$$\hat{\mathbf{v}}_1 = \frac{d\hat{\mathbf{x}}_1}{d\hat{t}} \quad \hat{\mathbf{v}}_2 = \frac{d\hat{\mathbf{x}}_2}{d\hat{t}} \quad \hat{\Delta\mathbf{x}} = \hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2 \quad \hat{\Delta\mathbf{v}} = \hat{\mathbf{v}}_1 - \hat{\mathbf{v}}_2. \tag{2}$$

Find the most general possible transform. In particular, a transform is suitable if it has the form above and every solution to (1) is transformed to a solution of (2).

Provide a physical interpretation for each of these degrees of freedom. That is, explain why *any* physically meaningful force between two particles must be invariant under these transforms, provided of course that its parameters are given suitable transform rules.

Finding the fully general set of transforms (there should be 10 degrees of freedom) and showing that they are suitable is worth one point. Showing that any suitable transform has this form (and as a result that there are not more than 10 degrees of freedom) is worth a second point. The physical intuition is worth a third point.

From $\mathbf{A}_2\mathbf{v}_1 + \mathbf{b}_2 = \hat{\mathbf{v}}_1 = \frac{d\hat{\mathbf{x}}_1}{d\hat{t}} = \frac{1}{\alpha_5}\mathbf{A}_1\frac{d\mathbf{x}_1}{dt} = \frac{1}{\alpha_5}\mathbf{A}_1\mathbf{v}_1$ *we see that* $\mathbf{b}_2 = \mathbf{0}$ *and* $\mathbf{A}_2 = \frac{1}{\alpha_5}\mathbf{A}_1$. *Similarly,* $\mathbf{A}_5 = \frac{1}{\alpha_5}\mathbf{A}_4$. *From* $\mathbf{A}_3\mathbf{F} + \mathbf{b}_3 = \hat{\mathbf{F}} = \hat{m_1}\frac{d\hat{\mathbf{v}}_1}{d\hat{t}} = \frac{1}{\alpha_5^2}\mathbf{A}_1(\alpha_3 m_1 + \beta_3)\frac{d\mathbf{v}_1}{dt} = \frac{\alpha_3}{\alpha_5^2}\mathbf{A}_1\mathbf{F} + \frac{\beta_3}{\alpha_5^2 m_1}\mathbf{A}_1\mathbf{F}$ *we conclude that* $\mathbf{A}_3 = \frac{\alpha_3}{\alpha_5^2}\mathbf{A}_1$, $\mathbf{b}_3 = \mathbf{0}$, *and* $\frac{\beta_3}{\alpha_5^2 m_1}\mathbf{A}_1\mathbf{F} = \mathbf{0}$. *Since* $\mathbf{A}_1$ *is nonsingular,* α_5 *is nonzero, and* m_1 *and* $\mathbf{F}$ *are variable, it must be that* $\beta_3 = 0$. *Similarly, one finds* $\mathbf{A}_3 = \frac{\alpha_4}{\alpha_5^2}\mathbf{A}_4$ *and* $\beta_4 = 0$ *and consequently* $\mathbf{A}_4 = \frac{\alpha_3}{\alpha_4}\mathbf{A}_1$.

Consider a steady state solution, where $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{F} = \Delta\mathbf{v} = \mathbf{0}$. *Then,* $\hat{\mathbf{v}}_1 = \hat{\mathbf{v}}_2 = \hat{\mathbf{F}} = \hat{\Delta\mathbf{v}} = \mathbf{0}$. *Substituting into (1) and (2) we find* $x_0 = \|\Delta\mathbf{x}\|$ *and* $\hat{x_0} = \|\hat{\Delta\mathbf{x}}\|$. *That is,* $\|\mathbf{A}_1\mathbf{x}_1 + \mathbf{b}_1 - \mathbf{A}_4\mathbf{x}_2 - \mathbf{b}_4\| = \|\hat{\Delta\mathbf{x}}\| = \alpha_6\|\Delta\mathbf{x}\| + \beta_6 = \alpha_6\|\mathbf{x}_1 - \mathbf{x}_2\| + \beta_6$. *Consider setups with* $\mathbf{x}_2 = \mathbf{0}$ *and* $\|\mathbf{A}_1\mathbf{x}_1 + \mathbf{b}_1 - \mathbf{b}_4\| = \alpha_6\|\mathbf{x}_1\| + \beta_6$. *From* $\mathbf{x}_1 \to \mathbf{0}$ *we see that* $\beta_6 = \|\mathbf{b}_1 - \mathbf{b}_4\|$. *Since* $\mathbf{A}_1$ *is invertible, we can choose a configuration with* $\mathbf{x}_1 = \mathbf{A}_1^{-1}(\mathbf{b}_4 - \mathbf{b}_1)$, *which gives us* $0 = \alpha_6\|\mathbf{A}_1^{-1}(\mathbf{b}_4 - \mathbf{b}_1)\| + \|\mathbf{b}_1 - \mathbf{b}_4\|$. *Since* $\alpha_6 > 0$, *we must conclude that* $\mathbf{b}_1 = \mathbf{b}_4$. *This reduces the system to* $\|\mathbf{A}_1\mathbf{x}_1\| = \alpha_6\|\mathbf{x}_1\|$. *Squaring both sides we have* $\mathbf{x}_1^T\mathbf{A}_1^T\mathbf{A}_1\mathbf{x}_1 = \alpha_6^2\mathbf{x}_1^T\mathbf{x}_1$ *or* $\mathbf{x}_1^T(\mathbf{A}_1^T\mathbf{A}_1 - \alpha_6^2\mathbf{I})\mathbf{x}_1 = \mathbf{0}$. *Since* $\mathbf{x}_1$ *is arbitrary,* $\mathbf{A}_1^T\mathbf{A}_1 = \alpha_6^2\mathbf{I}$. *Thus, we conclude that* $\mathbf{A}_1 = \alpha_6\mathbf{U}$, *where* $\mathbf{U}$ *is an orthogonal matrix. Since* $\mathbf{A}_1$ *has positive determinant and* $\alpha_6 > 0$, *we conclude that* $\mathbf{U}$ *is in fact a rotation matrix. If we repeat this with* $\mathbf{x}_1 = \mathbf{0}$ *instead, we find* $\mathbf{A}_4 = \alpha_6\mathbf{U}$, *which along with* $\mathbf{A}_4 = \frac{\alpha_3}{\alpha_4}\mathbf{A}_1$ *implies* $\alpha_3 = \alpha_4$.

Next, $\hat{\Delta\mathbf{x}} = \hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2 = (\alpha_6\mathbf{U}\mathbf{x}_1 + \mathbf{b}_1) - (\alpha_6\mathbf{U}\mathbf{x}_2 + \mathbf{b}_1) = \alpha_6\mathbf{U}(\mathbf{x}_1 - \mathbf{x}_2) = \alpha_6\mathbf{U}\Delta\mathbf{x}$ *and similarly* $\hat{\Delta\mathbf{v}} = \frac{\alpha_6}{\alpha_5}\mathbf{U}\Delta\mathbf{v}$. *Then,* $\hat{\mathbf{u}} = \frac{\hat{\Delta\mathbf{x}}}{\|\hat{\Delta\mathbf{x}}\|} = \frac{\alpha_6\mathbf{U}\Delta\mathbf{x}}{\|\alpha_6\mathbf{U}\Delta\mathbf{x}\|} = \mathbf{U}\frac{\hat{\Delta\mathbf{x}}}{\|\hat{\Delta\mathbf{x}}\|} = \mathbf{U}\mathbf{u}$.

We have so far now narrowed the transform possibilities to

$$\hat{\mathbf{x}}_1 = \alpha_6\mathbf{U}\mathbf{x}_1 + \mathbf{b}_1 \quad \hat{\mathbf{v}}_1 = \frac{\alpha_6}{\alpha_5}\mathbf{U}\mathbf{v}_1 \quad \hat{\mathbf{F}} = \frac{\alpha_3\alpha_6}{\alpha_5^2}\mathbf{U}\mathbf{F} \quad \hat{\mathbf{x}}_2 = \alpha_6\mathbf{U}\mathbf{x}_2 + \mathbf{b}_1 \quad \hat{\mathbf{v}}_2 = \frac{\alpha_6}{\alpha_5}\mathbf{U}\mathbf{v}_2$$

$$\hat{m_1} = \alpha_3 m_1 \quad \hat{m_2} = \alpha_3 m_2 \quad \hat{t} = \alpha_5 t + \beta_5 \quad \hat{x_0} = \alpha_6 x_0 \quad \hat{k_s} = \alpha_1 k_s + \beta_1$$

$$\hat{k_d} = \alpha_2 k_d + \beta_2 \quad \hat{\mathbf{u}} = \mathbf{U}\mathbf{u} \quad \hat{\Delta\mathbf{x}} = \alpha_6\mathbf{U}\Delta\mathbf{x} \quad \hat{\Delta\mathbf{v}} = \frac{\alpha_6}{\alpha_5}\mathbf{U}\Delta\mathbf{v}$$

At this point, we can take a look and see how close (1) and (2) are to matching up.

$$\begin{aligned}
\mathbf{0} &= \frac{\alpha_3\alpha_6}{\alpha_5^2}\mathbf{U}\mathbf{F} - \hat{\mathbf{F}} \\
&= -\frac{\alpha_3\alpha_6}{\alpha_5^2}\mathbf{U}k_s\left(\frac{\|\Delta\mathbf{x}\|}{x_0} - 1\right)\mathbf{u} - \frac{\alpha_3\alpha_6}{\alpha_5^2}\mathbf{U}k_d(\Delta\mathbf{v}\cdot\mathbf{u})\mathbf{u} + \hat{k_s}\left(\frac{\|\hat{\Delta\mathbf{x}}\|}{\hat{x_0}} - 1\right)\hat{\mathbf{u}} + \hat{k_d}(\hat{\Delta\mathbf{v}}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} \\
\mathbf{0} &= -\frac{\alpha_3\alpha_6}{\alpha_5^2}k_s\left(\frac{\|\Delta\mathbf{x}\|}{x_0} - 1\right)\mathbf{u} - \frac{\alpha_3\alpha_6}{\alpha_5^2}k_d(\Delta\mathbf{v}\cdot\mathbf{u})\mathbf{u} + \hat{k_s}\left(\frac{\|\alpha_6\mathbf{U}\Delta\mathbf{x}\|}{\alpha_6 x_0} - 1\right)\mathbf{u} + \hat{k_d}(\frac{\alpha_6}{\alpha_5}\mathbf{U}\Delta\mathbf{v}\cdot\mathbf{U}\mathbf{u})\mathbf{u} \\
\mathbf{0} &= -\frac{\alpha_3\alpha_6}{\alpha_5^2}k_s\left(\frac{\|\Delta\mathbf{x}\|}{x_0} - 1\right)\mathbf{u} - \frac{\alpha_3\alpha_6}{\alpha_5^2}k_d(\Delta\mathbf{v}\cdot\mathbf{u})\mathbf{u} + \hat{k_s}\left(\frac{\|\Delta\mathbf{x}\|}{x_0} - 1\right)\mathbf{u} + \hat{k_d}\frac{\alpha_6}{\alpha_5}(\Delta\mathbf{v}\cdot\mathbf{u})\mathbf{u} \\
&= -\left(\frac{\alpha_3\alpha_6}{\alpha_5^2}k_s - \hat{k_s}\right)\left(\frac{\|\Delta\mathbf{x}\|}{x_0} - 1\right)\mathbf{u} - \left(\frac{\alpha_3\alpha_6}{\alpha_5^2}k_d - \frac{\alpha_6}{\alpha_5}\hat{k_d}\right)(\Delta\mathbf{v}\cdot\mathbf{u})\mathbf{u}
\end{aligned}$$

From this and the fact that at any particular time, $\Delta \mathbf{x}$ and $\Delta \mathbf{v}$ could be anything (this equation must hold identically), one concludes $\beta_1 = \beta_2 = 0$, $\alpha_1 = \frac{\alpha_3 \alpha_6}{\alpha_5^2}$, and $\alpha_2 = \frac{\alpha_3}{\alpha_5}$. Since the equations are now equivelant, we know that the transforms that remain are suitable, and from the derivation above we know that no more degrees of freedom are possible in such a transform.

The degree of freedom α_6 corresponds to changing the units of length, α_5 corresponds to changing the units of time, and α_3 corresponds to changing the units of mass. The three degrees of freedom in $\mathbf{b}_1$ correspond to translation invariance. The three degrees of freedom in $\mathbf{U}$ correspond to rotation invariance, and the degree of freedom β_5 corresponds to time invariance. That is, the behavior of a spring does not depend on its location, its orientation (ignoring gravity), or when I examine the spring. Note that none of these depend in any particular way on the special properties of a spring. The first three are an artifact of how measurements are made, and the remaining seven are rather fundamental physical properties.

2. For each variable in (1), determine its SI units.

The quantities $\mathbf{x}_1$, $\mathbf{x}_2$, $\Delta \mathbf{x}$, and x_0 are positions, displacement, and length and have units of meters (m). The time t has units of seconds (s), and the masses m_1 and m_2 have units of kilograms (kg). $\mathbf{u}$ is a unit vector and is unitless. $\mathbf{v}_1$, $\mathbf{v}_2$, and $\Delta \mathbf{v}$ are velocities and have units $m\,s^{-1}$. $\mathbf{F}$ is a force and has units $kg\,m\,s^{-2}$. From the spring equation, we see that $\mathbf{F}$ has the same units k_s, so it also has units $kg\,m\,s^{-2}$. Finally, $\mathbf{F}$ has units of k_d times $\Delta \mathbf{v}$, so that k_d has units $kg\,s^{-1}$.

3. Show that the linear spring conserves mass, momentum, and angular momentum.

The mass is attached to the two particles, so it is trivially conserved. This is generally true of Lagrangian simulations. The total momentum of the system is $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2$. Then, $\frac{d\mathbf{p}}{dt} = m_1 \frac{d\mathbf{v}_1}{dt} + m_2 \frac{d\mathbf{v}_2}{dt} = \mathbf{F} - \mathbf{F} = \mathbf{0}$. Since momentum does not change in time, it is conserved. Consider the angular momentum about a fixed point $\mathbf{o}$. The angular momentum is $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 = (\mathbf{x}_1 - \mathbf{o}) \times \mathbf{p}_1 + (\mathbf{x}_2 - \mathbf{o}) \times \mathbf{p}_2$.

$$
\begin{aligned}
\frac{d\mathbf{L}}{dt} &= \frac{d}{dt}(\mathbf{x}_1 - \mathbf{o}) \times \mathbf{p}_1 + (\mathbf{x}_1 - \mathbf{o}) \times \frac{d\mathbf{p}_1}{dt} + \frac{d}{dt}(\mathbf{x}_2 - \mathbf{o}) \times \mathbf{p}_2 + (\mathbf{x}_2 - \mathbf{o}) \times \frac{d\mathbf{p}_2}{dt} \\
&= \mathbf{v}_1 \times \mathbf{p}_1 + (\mathbf{x}_1 - \mathbf{o}) \times \frac{d\mathbf{p}_1}{dt} + \mathbf{v}_2 \times \mathbf{p}_2 + (\mathbf{x}_2 - \mathbf{o}) \times \frac{d\mathbf{p}_2}{dt} \\
&= m_1 \mathbf{v}_1 \times \mathbf{v}_1 + (\mathbf{x}_1 - \mathbf{o}) \times \mathbf{F} + m_2 \mathbf{v}_2 \times \mathbf{v}_2 + (\mathbf{x}_2 - \mathbf{o}) \times (-\mathbf{F}) \\
&= \Delta \mathbf{x} \times \mathbf{F} \\
&= \|\Delta \mathbf{x}\| \mathbf{u} \times \left(-k_s \left(\frac{\|\Delta \mathbf{x}\|}{x_0} - 1 \right) \mathbf{u} - k_d (\Delta \mathbf{v} \cdot \mathbf{u}) \mathbf{u} \right) \\
&= \mathbf{0}
\end{aligned}
$$

4. Show that the energy for a (well-posed) spring is in general decreasing and find the condition a spring's parameters must satisfy to conserve energy. The potential energy of the spring is $U = \frac{1}{2}\frac{k_s}{x_0}(\|\Delta\mathbf{x}\| - x_0)^2$.

$$\begin{aligned}
E &= \frac{1}{2}m_1\|\mathbf{v}_1\|^2 + \frac{1}{2}m_2\|\mathbf{v}_1\|^2 + \frac{1}{2}\frac{k_s}{x_0}(\|\Delta\mathbf{x}\| - x_0)^2 \\
\frac{dE}{dt} &= m_1\mathbf{v}_1 \cdot \frac{d\mathbf{v}_1}{dt} + m_2\mathbf{v}_2 \cdot \frac{d\mathbf{v}_2}{dt} + \frac{k_s}{x_0}(\|\Delta\mathbf{x}\| - x_0)\frac{d}{dt}\|\Delta\mathbf{x}\| \\
&= \mathbf{v}_1 \cdot \mathbf{F} - \mathbf{v}_2 \cdot \mathbf{F} + \frac{k_s}{x_0}(\|\Delta\mathbf{x}\| - x_0)\frac{\Delta\mathbf{x}\cdot\Delta\mathbf{v}}{\|\Delta\mathbf{x}\|} \\
&= \Delta\mathbf{v} \cdot (\mathbf{F} + \frac{k_s}{x_0}(\|\Delta\mathbf{x}\| - x_0)\mathbf{u}) \\
&= \Delta\mathbf{v} \cdot \left(-k_s\left(\frac{\|\Delta\mathbf{x}\|}{x_0} - 1\right)\mathbf{u} - k_d(\Delta\mathbf{v}\cdot\mathbf{u})\mathbf{u} + k_s\left(\frac{\|\Delta\mathbf{x}\|}{x_0} - 1\right)\mathbf{u}\right) \\
&= -k_d(\Delta\mathbf{v}\cdot\mathbf{u})^2
\end{aligned}$$

If $k_d = 0$, then the spring does not lose energy. Otherwise, $k_d > 0$, and the energy change is negative unless $\Delta\mathbf{v}\cdot\mathbf{u} = \|\Delta\mathbf{x}\|^{-1}\frac{d}{dt}\|\Delta\mathbf{x}\|^2 = 0$. That is, a damped spring loses energy whenever the spring's length is changing.

5. Show that the center of mass of the system undergoes uniform translation. (That is, the center of mass moves through space with constant velocity.)

The center of mass is $= \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{m_1+m_2}$. Its change is $\frac{d}{dt} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1+m_2} = (m_1+m_2)^{-1}\mathbf{p}$, which is constant. Note that this is true whenever momentum is conserved.

6. Show that evolving (1) using forward Euler conserves mass and momentum but not angular momentum.

Mass is trivially conserved.

$$\begin{aligned}
\mathbf{p}^{n+1} &= \mathbf{p}_1^{n+1} + \mathbf{p}_2^{n+1} \\
&= m_1 \mathbf{v}_1^{n+1} + m_2 \mathbf{v}_2^{n+1} \\
&= m_1(\mathbf{v}_1^n + \Delta t\, m_1^{-1} \mathbf{F}^n) + m_2(\mathbf{v}_2^n - \Delta t\, m_2^{-1} \mathbf{F}^n) \\
&= (\mathbf{p}_1^n + \Delta t \mathbf{F}^n) + (\mathbf{p}_2^n - \Delta t \mathbf{F}^n) \\
&= \mathbf{p}_1^n + \mathbf{p}_2^n \\
&= \mathbf{p}^n \\
\mathbf{L}^{n+1} &= \mathbf{L}_1^{n+1} + \mathbf{L}_2^{n+1} \\
&= (\mathbf{x}_1^{n+1} - \mathbf{o}) \times \mathbf{p}_1^{n+1} + (\mathbf{x}_2^{n+1} - \mathbf{o}) \times \mathbf{p}_2^{n+1} \\
&= (\mathbf{x}_1^n - \mathbf{o} + \Delta t \mathbf{v}_1^n) \times (\mathbf{p}_1^n + \Delta t \mathbf{F}^n) + (\mathbf{x}_2^n - \mathbf{o} + \Delta t \mathbf{v}_2^n) \times (\mathbf{p}_2^n - \Delta t \mathbf{F}^n) \\
&= (\mathbf{x}_1^n - \mathbf{o} + \Delta t \mathbf{v}_1^n) \times \mathbf{p}_1^n + (\mathbf{x}_2^n - \mathbf{o} + \Delta t \mathbf{v}_2^n) \times \mathbf{p}_2^n + \Delta t(\Delta \mathbf{x}^n + \Delta t \Delta \mathbf{v}^n) \times \mathbf{F}^n \\
&= (\mathbf{x}_1^n - \mathbf{o}) \times \mathbf{p}_1^n + (\mathbf{x}_2^n - \mathbf{o}) \times \mathbf{p}_2^n + \Delta t^2 \Delta \mathbf{v}^n \times \mathbf{F}^n \\
&= \mathbf{L}^n + \Delta t^2 \Delta \mathbf{v}^n \times \mathbf{F}^n
\end{aligned}$$

Thus, angular momentum fails to be conserved when there is motion out of the spring direction. Note that the error goes away under refinement.