CS 205b / CME 306

Application Track

Homework 1

1. Use conservation of mass to show that the sum of the outward-facing area-weighted normals of a triangle mesh must be the zero vector.

   Let $\rho > 0$ and $\mathbf{u} \neq \mathbf{0}$ be a constant-density and constant-velocity fluid surrounding and flowing through this triangle mesh. The weak form of conservation of mass is

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_{\Omega} \rho \, dV &= -\int_{\partial\Omega} (\rho\mathbf{u}) \cdot dS \\
\frac{\partial}{\partial t}\int_{\Omega} dV &= -\int_{\partial\Omega} \mathbf{u} \cdot dS \\
0 &= -\mathbf{u} \cdot \int_{\partial\Omega} dS \\
0 &= -\mathbf{u} \cdot \sum_k \int_{T_k} dS \\
0 &= -\mathbf{u} \cdot \sum_k \mathbf{n}_k
\end{aligned}
$$

   where T_k are the triangles, and $\mathbf{n}_k$ is the area-weighted, outward-facing normal of T_k. Since $\mathbf{u}$ is arbitrary, it follows that

$$
\sum_k \mathbf{n}_k = \mathbf{0}.
$$

2. The strong form conservation of mass in an Eulerian frame can be written as $\rho_t + \rho_x u + \rho u_x = 0$. For each of the three terms:

   (a) Provide a physical description of what the term means,

   The ρ_t term describes how the density of a fixed point in space changes with time. The $u\rho_x$ term describes how mass advects (moves around) with the velocity field. The ρu_x term describes how mass compresses and expands in the velocity field.

(b) Describe a physical situation in which that term is identically zero in a region while the other two terms remain nonzero, and

If $\rho_t = 0$, then ρ is constant in time. Further, u and ρ must be spatially varying so that the second and third terms do not vanish. This would occur, for example, when air is forced through a nozzle and is in steady state.
If $u\rho_x = 0$, then either $u = 0$ or ρ_x. If $u = 0$ over a region of space, then $u_x = 0$, which makes the third term vanish as well. Thus, $\rho_x = 0$, and the density profile is spatially constant and time varying (to prevent the first term from vanishing). This situation would occur, for example, when a tire is being (slowly) filled with air.
If $\rho u_x = 0$, then $u_x = 0$, which implies that the velocity is spatially constant. That is, the fluid is simply advecting through space. Since ρ_x is nonzer, the density profile is spatially varying. This would occur, for example, if water whose sality is increasing over time flows at constant velocity through a pipe. Water becomes less dense as its sality increases, so $\rho_t < 0$.

(c) Show that the situation can actually occur by finding ρ and u such that the term is identically zero in the region $x, t \in [0, 1]$ while the other two terms are nonzero throughout the entire region.

The profile $\rho = x + 1$, $u = \frac{1}{x+1}$ makes only the first term vanish. The profile $\rho = t + 1$ and $u = -\frac{x+1}{t+1}$ makes only the second term vanish. The profile $\rho = x - t + 2$ and $u = 1$ makes only the third term vanish.

3. In this sequence of problems, we will construct a kernel function $W(\mathbf{x}, h)$ for use in the SPH method in 1D, 2D, and 3D.

(a) Since we would like $W(\mathbf{x}, h)$ to be symmetric about the origin, we take $W(\mathbf{x}, h) = c_d(h)f(\|x\|/h)$, where $c_d(h)$ is a normalization factor that depends on the dimension d and the radius of influence $h > 0$. The function $f(r)$ need not be defined for $r < 0$. Find $c_1(h)$, $c_2(h)$, and $c_3(h)$. (Hint: Use polar coordinates in 2D and spherical coordinates in 3D.)

For 1D, we have

$$\begin{aligned}
1 &= \int_{\mathbb{R}} W(\mathbf{x}, h) \\
&= 2\int_0^\infty c_1(h) f\left(\frac{r}{h}\right) dr \\
&= 2c_1(h)\int_0^\infty f\left(\frac{r}{h}\right) dr \\
&= 2c_1(h)\int_0^\infty f(u) h\, du \\
c_1(h) &= \left(2h\int_0^\infty f(r)\, dr\right)^{-1}
\end{aligned}$$

For 2D, we have

$$\begin{aligned}
1 &= \int_{\mathbb{R}^2} W(\mathbf{x}, h) \\
&= \int_0^{2\pi}\int_0^{\infty} r c_2(h) f\left(\frac{r}{h}\right) dr\, d\theta \\
&= c_2(h)\int_0^{2\pi} d\theta \int_0^{\infty} r f\left(\frac{r}{h}\right) dr \\
&= 2\pi c_2(h)\int_0^{\infty} r f\left(\frac{r}{h}\right) dr \\
&= 2\pi c_2(h)\int_0^{\infty} h u f(u) h\, du \\
c_2(h) &= \left(2\pi h^2 \int_0^{\infty} r f(r)\, dr\right)^{-1}
\end{aligned}$$

For 3D, we have

$$\begin{aligned}
1 &= \int_{\mathbb{R}^3} W(\mathbf{x}, h) \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r^2 \sin\phi\, c_3(h) f\left(\frac{r}{h}\right) dr\, d\phi\, d\theta \\
&= c_3(h)\int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\, d\phi \int_0^{\infty} r^2 f\left(\frac{r}{h}\right) dr \\
&= 4\pi c_3(h)\int_0^{\infty} r^2 f\left(\frac{r}{h}\right) dr \\
&= 2\pi c_3(h)\int_0^{\infty} (hu)^2 f(u) h\, du \\
c_3(h) &= \left(4\pi h^3 \int_0^{\infty} r^2 f(r)\, dr\right)^{-1}
\end{aligned}$$

(b) We would like the radius of influence of the kernel $W(\mathbf{x}, h)$ to be h. What conditions does this place on $f(r)$?

For any $\mathbf{x}$ where $\|\mathbf{x}\| > h$, we have $W(\mathbf{x}, h) = c_d(h) f(\|x\|/h) = 0$, which means $f(r) = 0$ if $r > 1$. For any $\mathbf{x}$ where $\|\mathbf{x}\| < h$, we have $W(\mathbf{x}, h) = c_d(h) f(\|x\|/h) > 0$, which means $f(r) > 0$ if $0 \le r < 1$. One may also take this to mean $f(1) = 0$ as well, and this will follow from the continuity requirement if one does not.

(c) We further require that $W(\mathbf{x}, h)$ have continuous second derivatives everywhere. What conditions does the continuity requirement place on $f(r)$? Be sure the kernel also satisfies this continuity requirement at the origin. (Hint: it is sufficient to look at 1D with $h = 1$.)

This certainly requires $f(r)$ to have a continuous second derivative everywhere. $W(x, 1) = c_d(1) f(|x|)$. Since $c_d(1)$ is a constant, it will not affect discontinuity and can be ignored.

Since $f(r)$ is already assumed to have continuous second derivatives $f(|x|)$ also will except possibly at $x = 0$.

$$\lim_{x \to 0^+} \frac{d}{dx} f(|x|) = \lim_{x \to 0^+} \frac{d}{dx} f(x) = f'(0)$$

$$\lim_{x \to 0^-} \frac{d}{dx} f(|x|) = \lim_{x \to 0^+} \frac{d}{dx} f(-x) = -f'(0)$$

Since $-f'(0) = f'(0)$, we need $f'(0) = 0$. (If this is not done, the kernel will come to a sharp "point" at the origin.)

$$\lim_{x \to 0^+} \frac{d^2}{d^2 x} f(|x|) = \lim_{x \to 0^+} \frac{d^2}{d^2 x} f(x) = f''(0)$$

$$\lim_{x \to 0^-} \frac{d^2}{d^2 x} f(|x|) = \lim_{x \to 0^-} \frac{d^2}{d^2 x} f(-x) = f''(0)$$

The second derivatives are already continuous.

(d) Find a suitable piecewise cubic function $f(r)$ defined for $r \geq 0$ that satisfies all of these requirements.

Since $f(r)$ has continuous second derivatives at $r = 1$, and $f(r)$ is identically zero for $r > 1$, it follows that $f(1) = f'(1) = f''(1) = 0$. Along with $f'(0) = 0$, this is four constraints. If a single cubic were used for $0 \leq r < 1$, then it must be that cubic is 0, which is not suitable. Thus, we will need to use (at least) two cubics to cover this region. The location of the transition between the two must be between 0 or 1, but it is otherwise somewhat arbitrary. We will choose $\frac{1}{2}$.

$$f(r) = \begin{cases} a_3 r^3 + a_2 r^2 + a_1 r + a_0 & 0 \leq r < \frac{1}{2} \\ b_3 r^3 + b_2 r^2 + b_1 r + b_0 & \frac{1}{2} < r < 1 \\ 0 & r \geq 1 \end{cases}$$

The constraints $f(1) = f'(1) = f''(1) = 0$ and $f'(0) = 0$ simplify this to

$$f(r) = \begin{cases} a_3 r^3 + a_2 r^2 + a_0 & 0 \leq r < \frac{1}{2} \\ b_3 (1 - r)^3 & \frac{1}{2} < r < 1 \\ 0 & r \geq 1 \end{cases}$$

Continuity of $f\left(\frac{1}{2}\right)$, $f'\left(\frac{1}{2}\right)$, and $f''\left(\frac{1}{2}\right)$ yield the constraints $\frac{1}{8}a_3 + \frac{1}{4}a_2 + a_0 = \frac{1}{8}b_3$, $\frac{3}{4}a_3 + a_2 = -\frac{3}{4}b_3$, and $3a_3 + 2a_2 = 3b_3$. Solving these gives

$$f(r) = \begin{cases} 6a_0 r^3 - 6a_0 r^2 + a_0 & 0 \leq r < \frac{1}{2} \\ 2a_0 (r - 1)^3 & \frac{1}{2} < r < 1 \\ 0 & r \geq 1 \end{cases}$$

Since we must normalize this function anyway, choose $a_0 = 0$.

$$f(r) = \begin{cases} 6r^3 - 6r^2 + 1 & 0 \leq r < \frac{1}{2} \\ 2(1 - r)^3 & \frac{1}{2} < r < 1 \\ 0 & r \geq 1 \end{cases}$$

(e) Evaluate $c_1(h)$, $c_2(h)$, and $c_3(h)$.

$$
\begin{aligned}
\int_0^\infty f(r)\,dr &= \int_0^{\frac{1}{2}} 6r^3 - 6r^2 + 1\,dr + \int_{\frac{1}{2}}^1 2(1-r)^3\,dr \\
&= \left[\frac{3}{2}r^4 - 2r^3 + r\right]_0^{\frac{1}{2}} + \left[-\frac{1}{2}(1-r)^4\right]_{\frac{1}{2}}^1 \\
&= \frac{3}{32} - \frac{1}{4} + \frac{1}{2} + \frac{1}{32} = \frac{3}{8} \\
c_1(h) &= \frac{4}{3h}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\infty r f(r)\,dr &= \int_0^{\frac{1}{2}} 6r^4 - 6r^3 + r\,dr + \int_{\frac{1}{2}}^1 2r(1-r)^3\,dr \\
&= \int_0^{\frac{1}{2}} 6r^4 - 6r^3 + r\,dr + \int_0^{\frac{1}{2}} 2(1-r)r^3\,dr \\
&= \int_0^{\frac{1}{2}} 4r^4 - 4r^3 + r\,dr \\
&= \left[\frac{4}{5}r^5 - r^4 + \frac{1}{2}r^2\right]_0^{\frac{1}{2}} \\
&= \frac{1}{40} - \frac{1}{16} + \frac{1}{8} = \frac{7}{80} \\
c_2(h) &= \frac{40}{7\pi h^2}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\infty r^2 f(r)\,dr &= \int_0^{\frac{1}{2}} 6r^5 - 6r^4 + r^2\,dr + \int_{\frac{1}{2}}^1 2r^2(1-r)^3\,dr \\
&= \int_0^{\frac{1}{2}} 6r^5 - 6r^4 + r^2\,dr + \int_0^{\frac{1}{2}} 2(1-r)^2 r^3\,dr \\
&= \int_0^{\frac{1}{2}} 8r^5 - 10r^4 + 2r^3 + r^2\,dr \\
&= \left[\frac{4}{3}r^6 - 2r^5 + \frac{1}{2}r^4 + \frac{1}{3}r^3\right]_0^{\frac{1}{2}} \\
&= \frac{1}{48} - \frac{1}{16} + \frac{1}{32} + \frac{1}{24} = \frac{1}{32} \\
c_3(h) &= \frac{8}{\pi h^3}
\end{aligned}
$$