CS 205b / CME 306

Application Track

Homework 3

1. Consider a 1D discretization with $\Delta x = \frac{1}{3}$ and the nine grid values $\rho_0 = 2$, $\rho_1 = 5$, $\rho_2 = 3$, $\rho_3 = 1$, $\rho_4 = -2$, $\rho_5 = -1$, $\rho_6 = 0$, $\rho_7 = 0$, $\rho_8 = 0$. Let the locations of these grid values be $x_0 = 0$, $x_1 = \Delta x$, $x_2 = 2\Delta x$, etc.

   (a) Construct the divided difference table for the data. Note that you will need to use Δx to construct this table. The first level of the table should consist of the ten values given above, and there should be three additional levels above it. Thus, your table should consist of $9 + 8 + 7 + 6$ entries.

			22.5		*-4.5*		*22.5*		*-18*		*-4.5*		*4.5*			
		-22.5		*0*		*-4.5*		*18*		*0*		*-4.5*		*0*		
	9		*-6*		*-6*		*-9*		*3*		*3*		*0*		*0*	
2		*5*		*3*		*1*		*-2*		*-1*		*0*		*0*		*0*
ρ_0		ρ_1		ρ_2		ρ_3		ρ_4		ρ_5		ρ_6		ρ_7		ρ_8

   (b) Assume information is flowing to the right ($u > 0$). For each of the positions x_3, x_4, and x_5, use third order HJ ENO to compute a Newton polynomial at that position. Call these polynomials $P_3^r(x)$, $P_4^r(x)$, and $P_5^r(x)$. You should leave your polynomials in the form of a Newton polynomial.

   For ρ_3, follow the sequence LLR. This gives the polynomial

$$P_3^r(x) = 1 - 6\left(x - \frac{3}{3}\right) - 4.5\left(x - \frac{3}{3}\right)\left(x - \frac{2}{3}\right)\left(x - \frac{1}{3}\right)$$

   For ρ_4, follow the sequence LLL. This gives the polynomial

$$P_4^r(x) = -2 - 9\left(x - \frac{4}{3}\right) - 4.5\left(x - \frac{4}{3}\right)\left(x - \frac{3}{3}\right) - 4.5\left(x - \frac{4}{3}\right)\left(x - \frac{3}{3}\right)\left(x - \frac{2}{3}\right)$$

   For ρ_5, follow the sequence LRR. This gives the polynomial

$$P_5^r(x) = -1 + 3\left(x - \frac{5}{3}\right) - 4.5\left(x - \frac{5}{3}\right)\left(x - \frac{4}{3}\right)\left(x - \frac{6}{3}\right)$$

(c) These polynomials are constructed to be interpolating polynomials. Show that $P_4^r(x)$ is in fact an interpolating polynomial.

$$
\begin{aligned}
P_4^r(x) &= -2 - 9\left(x - \frac{4}{3}\right) - 4.5\left(x - \frac{4}{3}\right)\left(x - \frac{3}{3}\right) - 4.5\left(x - \frac{4}{3}\right)\left(x - \frac{3}{3}\right)\left(x - \frac{2}{3}\right) \\
P_4^r(x_1) &= -2 - 9\left(\frac{1}{3} - \frac{4}{3}\right) - 4.5\left(\frac{1}{3} - \frac{4}{3}\right)\left(\frac{1}{3} - \frac{3}{3}\right) - 4.5\left(\frac{1}{3} - \frac{4}{3}\right)\left(\frac{1}{3} - \frac{3}{3}\right)\left(\frac{1}{3} - \frac{2}{3}\right) \\
&= -2 + 9 - 3 + 1 = 5 = \rho_1 \\
P_4^r(x_2) &= -2 - 9\left(\frac{2}{3} - \frac{4}{3}\right) - 4.5\left(\frac{2}{3} - \frac{4}{3}\right)\left(\frac{2}{3} - \frac{3}{3}\right) - 4.5\left(\frac{2}{3} - \frac{4}{3}\right)\left(\frac{2}{3} - \frac{3}{3}\right)\left(\frac{2}{3} - \frac{2}{3}\right) \\
&= -2 + 6 - 1 + 0 = 3 = \rho_2 \\
P_4^r(x_3) &= -2 + 3 - 0 - 0 = 1 = \rho_3 \\
P_4^r(x_4) &= -2 - 9\left(\frac{4}{3} - \frac{4}{3}\right) - 4.5\left(\frac{4}{3} - \frac{4}{3}\right)\left(\frac{4}{3} - \frac{3}{3}\right) - 4.5\left(\frac{4}{3} - \frac{4}{3}\right)\left(\frac{4}{3} - \frac{3}{3}\right)\left(\frac{4}{3} - \frac{2}{3}\right) \\
&= -2 - 0 - 0 - 0 = -2 = \rho_4
\end{aligned}
$$

(d) Assume instead that information is flowing to the left ($u < 0$). Use third order HJ ENO to compute the polynomials $P_3^l(x)$, $P_4^l(x)$, and $P_5^l(x)$. You should leave your polynomials in the form of a Newton polynomial.

For ρ_3, follow the sequence RLL. This gives the polynomial

$$P_3^l(x) = 1 - 9\left(x - \frac{3}{3}\right) - 4.5\left(x - \frac{3}{3}\right)\left(x - \frac{4}{3}\right) - 4.5\left(x - \frac{3}{3}\right)\left(x - \frac{4}{3}\right)\left(x - \frac{2}{3}\right)$$

For ρ_4, follow the sequence RRR. This gives the polynomial

$$P_4^l(x) = -2 + 3\left(x - \frac{4}{3}\right) - 4.5\left(x - \frac{4}{3}\right)\left(x - \frac{5}{3}\right)\left(x - \frac{6}{3}\right)$$

For ρ_5, follow the sequence RLR. This gives the polynomial

$$P_5^l(x) = -1 + 3\left(x - \frac{5}{3}\right) - 4.5\left(x - \frac{5}{3}\right)\left(x - \frac{6}{3}\right)\left(x - \frac{4}{3}\right)$$

(e) Above you computed six Newton polynomials. They should all look distinct, but they are not all distinct polynomials. Which polynomials are actually equal and why? You should not expand out the polynomals to answer this question.

Since the polynomials interpolate the data, they will the equal if they interpolate the same data. Thus, $P_3^r(x) = P_4^r(x) = P_3^l(x)$ and $P_5^r(x) = P_4^l(x) = P_5^l(x)$. Since $P_3^r(0) = 8$ and $P_4^l(0) = 14$, these two sets of polynomials are distinct.

2. There are multiple second order Runge Kutta schemes that one might use to evolve $x' = f(x)$. The classical one (and the one I am referring to when I write RK2) is $x^{n+1/2} = x^n + \frac{1}{2}\Delta t f(x^n)$, $x^{n+1} = x^n + \Delta t f(x^{n+1/2})$. Another second order Runge Kutta method is is TVD RK2, which has the form $\hat{x}^{n+1} = x^n + \Delta t f(x^n)$, $x^{n+2} = \hat{x}^{n+1} + \Delta t f(\hat{x}^{n+1})$, $x^{n+1} = \frac{1}{2}(x^n + x^{n+2})$.

(a) Show that these two schemes are in fact distinct schemes.

Let $f(x) = x^2$, $\Delta t = 1$, $x^n = 1$. Then, RK2 gives $x^{n+1/2} = \frac{3}{2}$ and $x^{n+1} = \frac{13}{4}$. From TVD RK2 we get $\hat{x}^{n+1} = 2$, $x^{n+2} = 6$, $x^{n+1} = 4$. Since the two schemes give different results, they must be different.

(b) In homework 2, question 3b, you expressed the update rule for a time integration scheme applied to $x' = \lambda x$ (complex λ) in the form $x^{n+1} = Cx^n$, where C is a complex number that depends only only the value of $\lambda\Delta t$. Compute the expression for C for both of these schemes. Let C_2 be the one you computed for RK2.

In this case, $f(x) = \lambda x$. RK2 gives

$$\begin{aligned} x^{n+1/2} &= x^n + \frac{1}{2}\Delta t \lambda x^n \\ x^{n+1} &= x^n + \Delta t \lambda (x^n + \frac{1}{2}\Delta t \lambda x^n) \\ &= x^n + \Delta t \lambda x^n + \frac{1}{2}(\Delta t \lambda)^2 x^n \\ C_2 = C &= 1 + \Delta t \lambda + \frac{1}{2}(\Delta t \lambda)^2 \end{aligned}$$

and TVD RK2 gives

$$\begin{aligned} \hat{x}^{n+1} &= x^n + \Delta t \lambda x^n \\ x^{n+2} &= (x^n + \Delta t \lambda x^n) + \Delta t \lambda (x^n + \Delta t \lambda x^n) \\ &= x^n + 2\Delta t \lambda x^n + (\Delta t \lambda x^n)^2 x^n \\ x^{n+1} &= \frac{1}{2}(x^n + x^n + 2\Delta t \lambda x^n + (\Delta t \lambda x^n)^2 x^n) \\ &= x^n + \Delta t \lambda x^n + \frac{1}{2}(\Delta t \lambda x^n)^2 x^n \\ C &= 1 + \Delta t \lambda + \frac{1}{2}(\Delta t \lambda)^2 \end{aligned}$$

(c) Use this to argue that the two schemes have identical stability plots. You do not need to construct the stability plots.

The stability of any particular region was determined based on the truth of $|C| < 1$. Since these schemes agree on C everywhere, their stability plots will agree everywhere.

(d) Let C_1 be the expression for C that is obtained for forward Euler, C_3 the expression obtained for TVD RK3, and C_4 the value obtained for RK4. It is okay to read off C_1 from the answer key to the assignment where you computed this, but you will need to derive C_3 and C_4.

As before, $C_1 = 1 + \Delta t\lambda$*. For TVD RK3,*

$$
\begin{aligned}
\hat{x}^{n+1} &= x^n + \Delta t\lambda x^n \\
x^{n+2} &= \hat{x}^{n+1} + \Delta t\lambda \hat{x}^{n+1} \\
&= x^n + \Delta t\lambda x^n + \Delta t\lambda(x^n + \Delta t\lambda x^n) \\
&= x^n + 2\Delta t\lambda x^n + (\Delta t\lambda)^2 x^n \\
x^{n+1/2} &= \frac{3}{4}x^n + \frac{1}{4}x^{n+2} \\
&= \frac{3}{4}x^n + \frac{1}{4}(x^n + 2\Delta t\lambda x^n + (\Delta t\lambda)^2 x^n) \\
&= x^n + \frac{1}{2}\Delta t\lambda x^n + \frac{1}{4}(\Delta t\lambda)^2 x^n \\
x^{n+3/2} &= x^{n+1/2} + \Delta t\lambda x^{n+1/2} \\
&= (x^n + \frac{1}{2}\Delta t\lambda x^n + \frac{1}{4}(\Delta t\lambda)^2 x^n) + \Delta t\lambda(x^n + \frac{1}{2}\Delta t\lambda x^n + \frac{1}{4}(\Delta t\lambda)^2 x^n) \\
&= x^n + \frac{3}{2}\Delta t\lambda x^n + \frac{3}{4}(\Delta t\lambda)^2 x^n + \frac{1}{4}(\Delta t\lambda)^3 x^n \\
x^{n+1} &= \frac{1}{3}x^n + \frac{2}{3}x^{n+3/2} \\
x^{n+1} &= \frac{1}{3}x^n + \frac{2}{3}(x^n + \frac{3}{2}\Delta t\lambda x^n + \frac{3}{4}(\Delta t\lambda)^2 x^n + \frac{1}{4}(\Delta t\lambda)^3 x^n) \\
x^{n+1} &= x^n + \Delta t\lambda x^n + \frac{1}{2}(\Delta t\lambda)^2 x^n + \frac{1}{6}(\Delta t\lambda)^3 x^n \\
C_3 &= 1 + \Delta t\lambda + \frac{1}{2}(\Delta t\lambda)^2 + \frac{1}{6}(\Delta t\lambda)^3
\end{aligned}
$$

Finally, for RK4 we get

$$\begin{aligned}
k_1 &= \lambda x^n \\
k_2 &= \lambda(x^n + \frac{1}{2}\Delta t \lambda x^n) \\
&= \lambda x^n + \frac{1}{2}\Delta t \lambda^2 x^n \\
k_3 &= \lambda(x^n + \frac{1}{2}\Delta t(\lambda x^n + \frac{1}{2}\Delta t \lambda^2 x^n)) \\
&= \lambda x^n + \frac{1}{2}\Delta t \lambda^2 x^n + \frac{1}{4}\Delta t^2 \lambda^3 x^n \\
k_4 &= \lambda(x^n + \Delta t(\lambda x^n + \frac{1}{2}\Delta t \lambda^2 x^n + \frac{1}{4}\Delta t^2 \lambda^3 x^n)) \\
&= \lambda x^n + \Delta t \lambda(\lambda x^n + \frac{1}{2}\Delta t \lambda^2 x^n + \frac{1}{4}\Delta t^2 \lambda^3 x^n) \\
&= \lambda x^n + \Delta t \lambda^2 x^n + \frac{1}{2}\Delta t^2 \lambda^3 x^n + \frac{1}{4}\Delta t^3 \lambda^4 x^n \\
x^{n+1} &= x^n + \frac{1}{6}\Delta t(k_1 + 2k_2 + 2k_3 + k_4) \\
x^{n+1} &= x^n + \Delta t \lambda x^n + \frac{1}{2}(\Delta t \lambda)^2 x^n + \frac{1}{6}(\Delta t \lambda)^3 x^n + \frac{1}{24}(\Delta t \lambda)^4 x^n \\
C_4 &= 1 + \Delta t \lambda + \frac{1}{2}(\Delta t \lambda)^2 + \frac{1}{6}(\Delta t \lambda)^3 + \frac{1}{24}(\Delta t \lambda)^4
\end{aligned}$$

(e) We could continue in this way using an explicit n-order scheme to derive C_n. What is C_∞ and why?

The solution the the differential equation is $x = e^{\lambda t}$*. If* $x^n = e^{\lambda t}$ *then* $x^{n+1} = e^{\lambda(t+\Delta t)} = e^{\lambda \Delta t} e^{\lambda t} = e^{\lambda \Delta t} x^n$*, so that*

$$C_\infty = e^{\lambda \Delta t} = 1 + \Delta t \lambda + \frac{1}{2}(\Delta t \lambda)^2 + \frac{1}{6}(\Delta t \lambda)^3 + \frac{1}{24}(\Delta t \lambda)^4 + \cdots .$$

(f) What does the stability region for C_∞ look like? You should work out the stability region analytically. You do not need to generate a stability plot for it.

Let $\lambda = a + bi$*.* $|C_\infty| = |e^{\lambda \Delta t}| = |e^{(a+bi)\Delta t}| = |e^{a\Delta t}||e^{b\Delta t i}| = e^{a\Delta t}$*.* $|C_\infty| < 1$ *and* $\Delta t > 0$ *implies* $a < 0$*. Thus, the stability region is the region where* $Re(\lambda) < 0$*, or the entire left half-plane.*