CS 205b / CME 306

Application Track

Homework 3

1. Consider a 1D discretization with $\Delta x = \frac{1}{3}$ and the nine grid values $\rho_0 = 2$, $\rho_1 = 5$, $\rho_2 = 3$, $\rho_3 = 1$, $\rho_4 = -2$, $\rho_5 = -1$, $\rho_6 = 0$, $\rho_7 = 0$, $\rho_8 = 0$. Let the locations of these grid values be $x_0 = 0$, $x_1 = \Delta x$, $x_2 = 2\Delta x$, etc.

   (a) Construct the divided difference table for the data. Note that you will need to use Δx to construct this table. The first level of the table should consist of the ten values given above, and there should be three additional levels above it. Thus, your table should consist of $9 + 8 + 7 + 6$ entries.

   (b) Assume information is flowing to the right ($u > 0$). For each of the positions x_3, x_4, and x_5, use third order HJ ENO to compute a Newton polynomial at that position. Call these polynomials $P_3^r(x)$, $P_4^r(x)$, and $P_5^r(x)$. You should leave your polynomials in the form of a Newton polynomial.

   (c) These polynomials are constructed to be interpolating polynomials. Show that $P_4^r(x)$ is in fact an interpolating polynomial.

   (d) Assume instead that information is flowing to the left ($u < 0$). Use third order HJ ENO to compute the polynomials $P_3^l(x)$, $P_4^l(x)$, and $P_5^l(x)$. You should leave your polynomials in the form of a Newton polynomial.

   (e) Above you computed six Newton polynomials. They should all look distinct, but they are not all distinct polynomials. Which polynomials are actually equal and why? You should not expand out the polynomals to answer this question.

2. There are multiple second order Runge Kutta schemes that one might use to evolve $x' = f(x)$. The classical one (and the one I am referring to when I write RK2) is $x^{n+1/2} = x^n + \frac{1}{2}\Delta t f(x^n)$, $x^{n+1} = x^n + \Delta t f(x^{n+1/2})$. Another second order Runge Kutta method is is TVD RK2, which has the form $\hat{x}^{n+1} = x^n + \Delta t f(x^n)$, $x^{n+2} = \hat{x}^{n+1} + \Delta t f(\hat{x}^{n+1})$, $x^{n+1} = \frac{1}{2}(x^n + x^{n+2})$.

   (a) Show that these two schemes are in fact distinct schemes.

   (b) In homework 2, question 3b, you expressed the update rule for a time integration scheme applied to $x' = \lambda x$ (complex λ) in the form $x^{n+1} = Cx^n$, where C is a complex number that depends only only the value of $\lambda \Delta t$. Compute the expression for C for both of these schemes. Let C_2 be the one you computed for RK2.

   (c) Use this to argue that the two schemes have identical stability plots. You do not need to construct the stability plots.

(d) Let C_1 be the expression for C that is obtained for forward Euler, C_3 the expression obtained for TVD RK3, and C_4 the value obtained for RK4. It is okay to read off C_1 from the answer key to the assignment where you computed this, but you will need to derive C_3 and C_4.

(e) We could continue in this way using an explicit n-order scheme to derive C_n. What is C_∞ and why?

(f) What does the stability region for C_∞ look like? You should work out the stability region analytically. You do not need to generate a stability plot for it.