CS 205b / CME 306

Application Track

Homework 3

1. A simple spring between particles at $\mathbf{x}_1$ and $\mathbf{x}_2$ in 3D can be defined by the equations

$$\mathbf{F} = m_1 \frac{d\mathbf{v}_1}{dt} \quad \mathbf{F} = -m_2 \frac{d\mathbf{v}_2}{dt} \quad \mathbf{u} = \frac{\Delta\mathbf{x}}{\|\Delta\mathbf{x}\|} \quad \mathbf{F} = -k_s\left(\frac{\|\Delta\mathbf{x}\|}{x_0} - 1\right)\mathbf{u} - k_d(\Delta\mathbf{v}\cdot\mathbf{u})\mathbf{u}$$
$$\mathbf{v}_1 = \frac{d\mathbf{x}_1}{dt} \quad \mathbf{v}_2 = \frac{d\mathbf{x}_2}{dt} \quad \Delta\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2 \quad \Delta\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2. \tag{1}$$

We would like to examine transformations under which these equations are invariant. That is, consider the new quantities obtained by applying transforms of the form

$$\begin{array}{lll}
\hat{\mathbf{x}}_1 = \mathbf{A}_1\mathbf{x}_1 + \mathbf{b}_1 & \hat{\mathbf{v}}_1 = \mathbf{A}_2\mathbf{v}_1 + \mathbf{b}_2 & \hat{\mathbf{F}} = \mathbf{A}_3\mathbf{F} + \mathbf{b}_3 \\
\hat{\mathbf{x}}_2 = \mathbf{A}_4\mathbf{x}_2 + \mathbf{b}_4 & \hat{\mathbf{v}}_2 = \mathbf{A}_5\mathbf{v}_2 + \mathbf{b}_5 & \\
\hat{m_1} = \alpha_3 m_1 + \beta_3 & \hat{m_2} = \alpha_4 m_2 + \beta_4 & \hat{t} = \alpha_5 t + \beta_5 \\
\hat{x_0} = \alpha_6 x_0 + \beta_6 & \hat{k_s} = \alpha_1 k_s + \beta_1 & \hat{k_d} = \alpha_2 k_d + \beta_2 \\
\hat{\Delta\mathbf{x}} = \hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2 & \hat{\Delta\mathbf{v}} = \hat{\mathbf{v}}_1 - \hat{\mathbf{v}}_2 & \hat{\mathbf{u}} = \dfrac{\hat{\Delta\mathbf{x}}}{\|\hat{\Delta\mathbf{x}}\|}
\end{array}$$

where $\mathbf{A}_i$ are non-singular matrices with positive determinant and α_i are positive. All of the matrices $\mathbf{A}_i$, vectors $\mathbf{b}_i$, and scalars α_i and β_i are constants. That is, they do not depend on t, $\mathbf{x}_1$, $\mathbf{x}_2$, or any of the other quantities that occur in (1). We also require that these transformed quantities also satisfy

$$\hat{\mathbf{F}} = \hat{m}\frac{d\hat{\mathbf{v}}_1}{d\hat{t}} \quad \hat{\mathbf{F}} = -\hat{m}\frac{d\hat{\mathbf{v}}_2}{d\hat{t}} \quad \hat{\mathbf{u}} = \frac{\hat{\Delta\mathbf{x}}}{\|\hat{\Delta\mathbf{x}}\|} \quad \hat{\mathbf{F}} = -\hat{k_s}\left(\frac{\|\hat{\Delta\mathbf{x}}\|}{\hat{x_0}} - 1\right)\hat{\mathbf{u}} - \hat{k_d}(\hat{\Delta\mathbf{v}}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}}$$
$$\hat{\mathbf{v}}_1 = \frac{d\hat{\mathbf{x}}_1}{d\hat{t}} \quad \hat{\mathbf{v}}_2 = \frac{d\hat{\mathbf{x}}_2}{d\hat{t}} \quad \hat{\Delta\mathbf{x}} = \hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2 \quad \hat{\Delta\mathbf{v}} = \hat{\mathbf{v}}_1 - \hat{\mathbf{v}}_2. \tag{2}$$

Find the most general possible transform. In particular, a transform is suitable if it has the form above and every solution to (1) is transformed to a solution of (2).

Provide a physical interpretation for each of these degrees of freedom. That is, explain why *any* physically meaningful force between two particles must be invariant under these transforms, provided of course that its parameters are given suitable transform rules.

Finding the fully general set of transforms (there should be 10 degrees of freedom) and showing that they are suitable is worth one point. Showing that any suitable transform has this form (and as a result that there are not more than 10 degrees of freedom) is worth a second point. The physical intuition is worth a third point.

2. For each variable in (1), determine its SI units.

3. Show that the linear spring conserves mass, momentum, and angular momentum.

4. Show that the energy for a (well-posed) spring is in general decreasing and find the condition a spring's parameters must satisfy to conserve energy. The potential energy of the spring is $U = \frac{1}{2}\frac{k_s}{x_0}(\|\Delta\mathbf{x}\| - x_0)^2$.

5. Show that the center of mass of the system undergoes uniform translation. (That is, the center of mass moves through space with constant velocity.)

6. Show that evolving (1) using forward Euler conserves mass and momentum but not angular momentum.