CS 205b / CME 306

Application Track

Homework 2

1. **ALE** An Eulerian formulation of conservation of mass uses control volumes that are fixed in space as material flows freely through the control volumes. A Lagrangian formulations uses control volumes that move with the material, so that material never flows into our out of the control volume. An ALE (Arbitrary Lagrangian Eulerian) formulation is somewhere between these two. Control volumes move around with a velocity $\mathbf{v}$, as material flows freely through them. The material has density ρ and velocity $\mathbf{u}$. The velocity field $\mathbf{v}$ of the observer and the velocity field $\mathbf{u}$ of the material being observed are independent and may vary in both space and time. In particular, control volumes can move around and change shape over time.

   (a) Adapt the derivation of the weak form for conservation of mass to the ALE case, where the control volume itself also moves around based on a velocity field $\mathbf{v}$.

   (b) Show that in the special case that $\mathbf{v} = \mathbf{0}$, the weak form derived for the Eulerian case is recovered.

   (c) Show what equation is obtained in the special case that $\mathbf{v} = \mathbf{u}$. Give a physical explanation for why this equation corresponds to a Lagrangian formulation of conservation of mass in weak form.

   (d) Convert the ALE weak form of conservation of mass into strong form. Be careful when moving the time derivative inside the integration. Show that this matches what was obtained using the Eulerian formulation.

2. **Duhamel's Principle** Consider the two ordinary differential equations $x' = \lambda x$ and $y' = \lambda y + \gamma$, where x, y, λ, and γ are all complex numbers.

   (a) Find analytic solutions to the recurrences $r_{n+1} = \alpha r_n$ and $s_{n+1} = \alpha s_n + \beta$. Be careful of special cases.

   (b) For which α and β is r_n bounded but s_n unbounded?

   (c) For which α and β is s_n bounded but r_n unbounded?

   (d) Show that for trapezoid rule, the update rule for x_n has the same form as the recurrence r_n, and the update rule for y_n has the same form as recurrence s_n. Also show that the expressions for α are the same and do not depend on γ.

   (e) What do you conclude about the dependence of the stability of trapezoid rule on the inhomogeneous term γ? Would the conclusion change much if backward Euler or forward Euler were being studied instead?

3. **Plotting Stability** Note this problem contains a small programming component. The lecture notes show stability plots for forward Euler (FE), backward Euler (BE), trapezoid rule (TR), second order Runge-Kutta (RK2), third order Runge-Kutta (RK3), and fourth order Runge-Kutta (RK4). These plots were obtained by considering the equation $y' = \lambda y$, where λ is complex.

   (a) State the update rules for FE, BE, TR, and RK2 when applied to $y' = f(y)$.

   (b) Let $f(y) = \lambda y$, so that the differential equation is $y' = \lambda y$. When solved for y_{n+1} in terms of y_n with, the update rule should have the form $y_{n+1} = C y_n$, where C is a complex number that depends only on λt. Find C for FE, BE, TR, and RK2.

   (c) What must be true of C for a method to be stable for a given choice of Δt and λ?

   (d) Let $\Delta t = 1$ and sample the complex plain in the region $-3 \leq \text{Re}(\lambda) \leq 3$ and $-3 \leq \text{Im}(\lambda) \leq 3$, determining for each value of λ whether the scheme will be stable. Use white to indicate unstable and a distinct color (not black) to indicate stable. Add black axes to the images (two lines is fine), making sure that the axes are on top of everything else. It is recommended that you use matlab or octave for this assignment, though a solution using C++ and ImageMagick is also acceptable. For each of FE, BE, TR, and RK2, you should submit (on paper) the image obtained and the source code used to construct it (even if the four programs are nearly identical). The images should closely match the ones in the lecture notes.